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Supplement to "Some Conjectures Concerning Triangular Numbers"

Bruce Brandt

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Introduction

In a previous paper (1), I stated many conjectures about triangular numbers. Since submitting that paper I have discovered many more results, including generalizations, which are presented here.

1. Patterns like s_k and a_k exist not only for t_n squares or anti-squares, but also for any t_n where $\sigma(t_n)$ or $s(t_n)$ is fixed, $\alpha(j)$ being j minus the nearest anti-square, i.e., number of the form m^2+m (α is undefined on squares). The greatest chance of not being an exception is when t_n is large and $|\sigma(t_n)|$ or $|\alpha(t_n)|$ is small.

2. Some of the patterns generalize in some form to some other pairs of integer-valued quadratic functions besides $m(m+1)/2$ and m^2. For example, there are patterns in the sign and parity of σ and τ on the pentagonal numbers.

3. The values of σ on the positive t_n which are 0 or +1 form this pattern: 0, -1, 1, -1, 0, -1, 1, -1, 0, -1, 1, -1, 0, -1, 1, -1, 0, -1, 1, -1, 0, -1, 1, -1, …

Furthermore, if $\sigma(t_n) = 1$ and t_m is the next higher triangular number such that $\sigma(t_m) = -1$, then the sum of $\sigma(t_k)$ for $t_n < t_k < t_m$ is 0. Example: there are no t_k between $t_n = 10$ and $t_m = 15$

4. I have discovered a better way of writing the algorithm for the signs of σ on the triangular numbers on page 24:

1. Start with a "1".
2 Repeat n-1 times the sequence of first 2a then 2b:
   a. Simultaneously replace every "1" with a "12121" and every "2" with a "1212121".
   b. Add an initial "1" and a final "1".
3. Simultaneously replace every "1" with a "122" and every "2" with a "1222". Then add a final "1".

In the case of the signs from 36 to 1225, the algorithm yields at each step:

1.		1					
2a.		1	2	1	2	1	
2b.	1	1	2	1	2	1	1

3. 12212212221221222122122 1

This algorithm is equivalent to, but better than, the one I wrote because step 2 (the only one I changed) is simpler, more symmetric, and more similar to other patterns in the paper.

5. I refer to generalizations to quadratic functions above. In fact, some patterns can also be found by considering $\gamma(n)$, defined as $n - m^3$, where m^3 is the nearest cube to n, on the domain of the tetrahedronal numbers, defined as $T_n = n(n+1)(n+2)/6$. This suggests that the phenomena apply to polynomials in general.

For example, the values of $\gamma(T_{n+1}) - \gamma(T_n)$, for $\gamma(T_{n+1}) > 0$ and $\gamma(T_n) < 0$ are, in order, 15,28,45,78, 105,136,171,231,276,....

These are all triangular numbers. Each one is the 2nd or 3rd triangular number after the previous one.

On the other hand, the first three values of $\gamma(T_n) - \gamma(T_{n+1})$, for $\gamma(T_n) > 0$ and $\gamma(T_{n+1}) < 0$ are, in order, 9,16,25. These, of course, are all squares. But the next five such values are 61,78,97,118,141. These are all of the form m^2-3. The next four values are all of the form m^2-9. In general, there are clumps of four or five values (except the first clump only consists of three), of the form m^2-3t_{k-1}, where t represents the triangular numbers as before and k is the rank of the clump. Within each clump, m progresses by 1 (e.g., 8^2-3, 9^2-3, 10^2-3, 11^2-3, 12^2-3 in the second clump) but m progresses by 3 between clumps (e.g., from 5 to 8 between the first and second clumps).

As long as the signs of γ alternate regularly on the T_n, we have (in the second to last paragraph) each value being the 2nd triangular number after the last and (in the last paragraph) a single clump, and as long as that happens these patterns are reducible to algebra. But the pattern of "2"'s and "3"'s (from the second to last paragraph) and the pattern of "4"'s and "5"'s (from the last paragraph) need to be, and apparently can be, generated by an algorithm like those we have already seen, and this is nontrivial.

6. I have proven the assertion on pages 24-25 that the values of $\tau(n^2)-\tau((n+1)^2)$, for $\tau(n^2) > 0$ and $\tau((n+1)^2) < 0$, are the positive even numbers, in order and without skipping. The key lemma is:

Lemma For $n > 0$, let t_m be the nearest triangular number to n^2. If $\tau(n^2) > 0$ and $\tau((n+1)^2) < 0$, the nearest triangular number to $(n+1)^2$ is t_{m+2}; otherwise it is t_{m+1}.

This Lemma can be proven with reasonably straightforward algebraic manipulation. With the Lemma it is easy to show that the values of $\tau(n^2)$-

† Independent contribution

‡ 13 27th Avenue S.E., Minneapolis, MN 55414-3101.

$\tau((n+1)^2)$ under the given conditions are always even and each value is exactly two more than the previous one.

A proposition similar to the above Lemma seems to be true of σ on the triangular numbers.

7. The sequence 3,7,10,14,17,20, ... , described before, has a simple formula. The nth member of the sequence is the closest integer to $(2+\sqrt{2})n$. Elements of the complementary set are the closest integer to $\sqrt{2}(n+1/2)$.

8. Let G_n be the sequence of geometric means between a triangular square and the next smaller, or next larger, triangular anti-square. The fractional part of $4G_n$ approaches the limit of $(3-\sqrt{2})/2 \approx .7928932188134525$. Not only do the empirical data suggest this, but I have convinced myself of it by manipulating the explicit formulas for triangular squares and triangular anti-squares, although the proof is not conclusive.

REFERENCE

1. Brandt, B. 1994. Some Conjectures Concerning Triangular Numbers. J. Minn. Acad. Sci. 58:21-25.