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The Theory of the Impulse Function for Oscillatory Mechanical and Electrical Systems

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To see beats with a stringed instrument use two identical strings. The thicker they are the better. Arrange them in such a way that they cross each other near their middle and that one end of the top one is raised up a little so that the two strings do not touch each other. Tune them to the same pitch. If they are within a few vibrations of each other the beats can be plainly seen where the strings cross. By changing the tension of one of the strings slightly the number of beats can be increased or decreased at will or the beats may be eliminated altogether.

THE THEORY OF THE IMPULSE FUNCTION FOR OSCILLATORY MECHANICAL AND ELECTRICAL SYSTEMS

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Several general methods are available for the representation of the solutions of the equations of oscillatory mechanical and electrical systems. In any particular problem, the choice of the method to be used is largely a matter of convenience. For those problems in which the characteristic frequencies and modes of oscillation are of direct interest, an expansion in terms of normal functions is to be preferred. If one is dealing with a system with continuously distributed physical parameters this leads to expansions in infinite series of functions such as the trigonometric functions (Fourier series and integrals), Bessel functions, Legendre functions, and so on.

For dealing with problems in which the physical data are given in the form of initial conditions, and particularly in cases involving externally applied forces, the method of expansion in normal functions is apt to become very involved, and the series obtained are often only asymptotic in character, if indeed they can be summed in any sense. Following the theory proposed by Heaviside, electrical circuits are often treated by considering that the external voltages applied to the system are resolvable into a sequence of "step-functions." The basic element in this analysis is the *unit function* H (t) which is defined by the properties

$$\begin{aligned} H(t) &= 0 \quad \text{if} \quad t < 0 \\ &= 1 \quad \text{if} \quad t \geqq 0 \end{aligned}$$

Mathematically this procedure may be considered as a resolution by means of Stieltje's integral rather than the Fourier integral. The unit function is taken as the integration element.

For the discussion of mechanical systems it seems more suggestive from the physical point of view to introduce the idea of impulsive forces. An elementary impulse is considered as represented by a force (or electromotive force) of very great magnitude which is applied to the system over only a very short interval of time. The time integral of the force is finite. The idealization of this definition leads one to the notion of the *delta function* $\delta(t)$ having the formal properties

$$t\ \delta(t) = 0$$
$$\int \delta(t)\ dt = 1$$

The range of integration is any finite interval containing the point $t=0$ in its interior. One can also characterize this function formally by the symbolic relation

$$\delta(t) = dH/dt.$$

Since functions with these properties can be included within the class of continuous differentiable functions only in the sense of limiting cases, they are difficult to incorporate directly into the theory of the differential equations governing mechanical and electrical systems. However, by employing the operational calculus based on the Laplace transformation theory a coherent and unified procedure can be built up which is of considerable value in practical applications.

We define the Laplace transform of a function f(t) by the relation

$$\varphi(p) = p \int_0^\infty e^{-pt}\ f(t)\ dt.$$

The transform of the function H(t) is then just unity, while the transform of δ (t) is the function p. On going over to the operational equations of motion, it is found that, provided the system is not considered to be devoid of mass or inductance elements, it is even possible to incorporate the initial conditions in the form of apparent forces of a very singular type associated with the operational form p^2.

As one of the consequences of Heaviside's procedure, it can be shown (Duhamel's theorem) that if the reaction of the system to an applied force or voltage of the form of the unit function H(t) is known, then its reaction to one of arbitrary form can be determined from the formula

$$x(t) = \frac{d}{dt}\left\{ \int_0^t x_H\ (t-T)\ E\ (T)\ dT \right\}$$

in which x(t) is the displacement function under the applied force E(t), while $x_H(t)$ is the displacement for an applied force of form H(t).

When making use of the analysis by means of impulse functions there exists a similar relation of the form

$$x(t) = \int_0^t x_o\ (t-T)\ E(T)\ dT$$

in which $x_o(t)$ is the displacement function under an applied force of the form δ (t). This is readily shown to be equivalent to the Heaviside formula, assuming that the function E(t) vanishes for $t < 0$, and the system starts from a quiescent condition at this instant.

For continuous systems both of these procedures can be generalized without difficulty, but the one depending on impulse functions becomes particularly useful. It turns out that its use leads one quite simply and directly to a type of analysis quite similar to the use of Green's functions in the solution of partial differential equations.

One of the most interesting applications which has yet been made of this analysis is to the theory of electromechanical systems. These have the peculiarity that, unlike oscillatory systems of either purely mechanical or electrical type, the matrix of the equation obtained for the amplitude functions is skew-symmetric, rather than symmetric. By employing a linear transformation of the operational equations of mixed character between displacement and force functions (or currents and emf's) one can throw them into symmetric form. This sets up a type of analogy between electrical and mechanical systems of exactly the type suggested by Firestone and by Hahnle. The application of the impulse functions to the solution then readily leads to a complete analysis of such systems. The complete details of this work will be published in due course elsewhere.

SOME EMISSION PHENOMENA OBSERVED ON COMMERCIAL PHOTOCELLS

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The author used for special photometric measurements a control circuit as suggested by RCA (1). The principle of the circuit is that a condenser is charged with direct current and discharged by a photocell that is exposed to light. The more light that falls upon the photocell the faster the condenser will discharge, and the shorter will be the operation interval of a relay introduced into this circuit.

If one uses this circuit for exposure control of a photographic process one must carefully select the photocell. Focussing and manipulation in the dark-room are frequently done by using safe-light. If a photocell sensitive to red or infra-red were used, the activation of the relay would be affected by this illumination in addition to the exposure proper. It is evident therefore that a photocell not sensitive to red or infra-red light must be used, such as a RCA 929 phototube, the threshold wavelength of which is approximately 630 mμ as indicated in the RCA pamphlet (1).

The author noticed that the operation interval used to expose the photographic emulsion was shorter when the focussing filter