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On Frege's Alleged Indispensability Argument

Pieranna Garavaso
University of Minnesota, Morris, garavapf@morris.umn.edu

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1. Introduction

The name "indispensability arguments" designates a family of arguments[1] meant to support mathematical realism, i.e., the view that there exist mind-independent mathematical objects. One of the simplest versions of this argument claims that we must accept the existence of mathematical entities because mathematical theories are indispensable to sciences such as physics, and mathematical statements refer to mathematical entities. The two main premises of this version of the argument are the claim of the indispensability of mathematics to the development of scientific theories and the claim that some ontological commitment follows from the use of mathematical statements which refer to mathematical entities. This ontological commitment is sometimes justified semantically, sometimes pragmatically.[2]

Putnam and Quine are commonly cited as the originators of this argument, but more and more often Gottlob Frege is credited with having been the first to employ this argument to

[1] See Maddy [1992], p. 275 and Colyvan [2001], pp. 1-19.

[2] Putnam cites Quine's portrayal of the pragmatic justification, see Putnam [1971], p. 347. Resnik defends a *Pragmatic Indispensability Argument* against Maddy's and Sober's criticisms of a *Confirmational Indispensability Argument*. Colyvan regards Resnik's pragmatic version as "a very powerful argument" which "doesn't rely on [Quine's] confirmational holism." See Resnik [1995], Maddy [1992], Sober [1993], and Colyvan [2001], pp. 13-15.

support mathematical Platonism.[3] This alleged indispensability argument, stated by Frege in section 91 of the *Grundgesetze der Aritmetik*, is the focus of this essay. After an overview of Colyvan's indispensability arguments, I deny that Frege's argument is an indispensability argument by showing that it lacks three common features of Colyvan's arguments.

2. Indispensability arguments

Colyvan mentions several indispensability arguments, but examines more closely three. Among the versions he mentions but does not discuss extensively, there is Resnik's pragmatic argument,[4] Gödel's version in "What is Cantor's Continuum Problem?"[5] and the version supposedly advanced by Frege in section 91 of *Grundgesetze*.

Here are the three main versions of the indispensability argument discussed by Colyvan:

Argument 1 (Scientific Indispensability Argument) *If apparent reference to some entity (or class of entities)   is indispensable to our best scientific theories, then we ought to believe in the existence of .*

Argument 2 (The Quine/Putnam Indispensability Argument)

> *1. We ought to have ontological commitment to all and only those entities that are indispensable to our best scientific theories.*

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[3] SeeMaddy [1992], p. 275, Balaguer [1998], p. 96, Colyvan [2001], pp. 8-9. Platonism differs from realism in requiring mathematical objects to be abstract entities. This distinction plays no role in my discussion; so, I treat Platonism and realism as synonymous.

[4] I do not discuss Resnik's argument here since Colyvan makes no connection between Resnik's version and the alleged version attributed to Frege. See Colyvan [2001], pp. 13-15.

[5] See Colyvan [2001], p. 9, and Gödel [1947], p. 521.

2. Mathematical entities are indispensable to our best scientific theories.

Therefore:

3. We ought to have ontological commitment to mathematical entities.[6]

Argument 3 (Semantic Indispensability Argument) *If apparent reference to some entity (or class of entities) is indispensable to our best semantic theories of natural (and scientific) language, then we ought to believe in the existence of . Abstracta are indispensable to our best semantic theory of natural (and scientific) language. Thus, we ought to believe in such abstracta.*[7]

The first argument is a succinct enthymeme: it lacks both the conclusion that we ought to believe in the existence of ξ, and the premise that apparent reference to some entity (or class of entities) ξ is indispensable to our best scientific theories. This version applies to any type of entity ξ ; to support mathematical realism, we must assume that “ ” denotes mathematical entities.

By using the expression “apparent reference,” Colyvan is not suggesting that reference to mathematical entities in mathematical statements is misleading or dispensable. Although the eliminability of *prima facie* reference to mathematical entities is a matter of contention between realists and anti-realists, here, Colyvan acknowledges a point admitted by both sides, i.e., that by using singular terms such as “three” and “nine” in statements such as “Three is the root square of nine,” we seem to refer directly to mathematical entities such as three and nine.

According to Colyvan, the third version is the weakest since semantics is less rigorous than the natural sciences to which the first two versions appeal. Moreover, the plausibility of the first premise of the second version, i.e., the claim that we ought to have ontological commitment

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[6] Colyvan states only this version in Colyvan [2003].

[7] See Colyvan [2001], pp. 7-16.

to all and only those entities that are indispensable to our best scientific theories, relies on the acceptance of two Quinean doctrines such as holism and naturalism.[8] Accordingly, this second version cannot be appealing to philosophers unwilling to accept Quine's assumptions.

3. The notion of indispensability.

What does "indispensable" mean in Colyvan's arguments?[9] Does it mean the same whether mathematics, or abstracta, or apparent reference to some entities are indispensable? Colyvan so defines "dispensable:"

> *An entity is dispensable to a theory iff the following two conditions hold:*
>
> *(1) There exists a modification of the theory in question resulting in a second theory with exactly the same observational consequences as the first, in which the entity in question is neither mentioned nor predicted.*
>
> *(2) The second theory must be preferable to the first.*[10]

Here, it is apparent that Colyvan shares Quine's naturalistic assumption: entities are indispensable to a scientific theory if their elimination causes the loss of some prized virtues of our preferable theories such as simplicity, elegance, or boldness.[11]

Other proponents and discussants of indispensability arguments appeal to different criteria. For example, Maddy describes Putnam's notion as stronger than Quine's and Colyvan's:

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[8] See Colyvan [2001], pp. 12-3.

[9] See Colyvan [2001], section 4.2, pp. 76-78.

[10] See Colyvan [2001], p. 77.

[11] Colyvan lists and explains "desiderata for 'good theories'" on pp. 78-79.

> Putnam takes the same thinking [as Quine's] somewhat further, emphasizing not only that mathematics simplifies physics, but that physics can't even be formulated without mathematics.[12]

Here, mathematics is not only a means for simplifying physics; it is essential for its formulation.

Putnam's criteria for indispensability rely on semantic considerations involving the meaning of statements in which numerical terms or quantification over numerical entities occur:

> So far I have been developing an argument for realism along roughly the following lines: quantification over mathematical entities is indispensable for science, both formal and physical; therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question.[13]

In Putnam's version of the indispensability argument, quantification over mathematical entities is the major criterion by which the indispensability of mathematics to science is measured.

In the passage just quoted, Putnam does not appeal to true quantified statements, but Resnik, in earlier versions of his rendition of Quine and Putnam's argument, and Sober, among others, mention the truth of mathematical statement in connection with indispensability:

> According to Quine and Putman, appealing to mathematical objects and mathematical truths figures indispensably in using mathematics in science, and as a consequence, we should consider mathematical objects to be no less real than scientific ones.[14]

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[12] Maddy [1990], p. 29.

[13] See Putnam [1975], p. 347.

[14] Resnik [1992], p. 115. Resnik seems not to believe that all indispensability arguments must assume (or imply) the truth of mathematical statements. In 1995, he stated: "[the *Pragmatic Indispensability Argument*] is weaker than the confirmational argument in not concluding that mathematics is true." See Resnik [1995], p. 170.

> According to this line of thinking [which leads to the conclusion that the empirical success of a scientific theory *confirms* the mathematical claims embedded within it] we have reason to believe that mathematical statements are *true*, and that the entities they quantify over *exist*, because mathematics is indispensable to empirical science.[15]

Sober's indispensability argument purports to derive the reasonableness of believing in the truth of mathematical statements and in the existence of the entities over which they quantify.

Another criterion often appealed to in arguing for the indispensability of mathematics to the sciences is the reference of mathematical terms and quantification. Field for example explains how such indispensability raises a "problem" for a nominalist like himself as follows:

> [O]ur ultimate account of what the world is really like must surely include a physical theory; and in developing physical theories one needs to use mathematics; and mathematics is full of such references to and quantifications over numbers, functions, sets, and the like. It would appear then that nominalism is not a position that can be reasonably maintained.[16]

Like Putnam, Field mentions quantification over mathematical entities, but before that, he cites reference to mathematical objects. So, for Field the use of singular terms to denote mathematical entities is a major problem for a nominalist revision of the use of mathematics in the sciences.

The notions of indispensability occurring in the above passages show how applications of, and appeals to, mathematics take different forms. This is significant for my discussion,

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[15] See Sober [1993], p. 35.

[16] See Field (1980), p. 1. For Maddy, reference is a necessary feature of the indispensability of mathematics as well, see Maddy (1990), p. 59, footnote 55.

because Frege's appeal to the applicability of mathematics ensues from a context that has nothing in common with the main preoccupation of these arguments, i.e., the scientific employment of mathematical statements. In order to take seriously Colyvan's claim that, in section 91 of the *Grundgesetze,* Frege presents an indispensability argument, we need a viable notion of indispensability which is different from the notions we have seen so far. Colyvan offers the following intuitive notion at the beginning of his discussion of indispensability arguments:

> How are we to understand the phrase "indispensable to our best scientific theory?" In particular, what does "indispensable" mean in this context? ...[Let us] take it to intuitively mean "couldn't get by without" or some such. In fact, whatever sense it is in which electrons, neutron stars, and viruses are indispensable to their theories will do.[17]

Supporters of indispensability arguments take mathematical entities to be as necessary as scientific entities to the relevant sciences. This notion mentions neither true mathematical statements nor the practices of referring to, or quantifying over, mathematical entities. It is thus useful for a comparison with Frege's argument for the latter appeals to none of the above. If Frege's argument is not an indispensability argument under this notion, it cannot be regarded as an indispensability argument under any of the other notions.

4. The main features of Colyvan's indispensability arguments

Colyvan's arguments are stated as valid deductive arguments. Thus, his claim that the indispensability argument is an inductive inference to the best explanation may seems

[17] See Colyvan (2001), p. 12.

surprising.[18] The confusion is soon cleared, by noticing that an inference to the best explanation supports the main premise in each of these arguments. We should accept the existence of the entities denoted by numerical terms or quantified over in mathematical statements because their existence explains more easily why we take these propositions as true, i.e., why we treat them as correct descriptions of actual entities and their properties. In discussion of inferences to the best explanations, the most controversial issue is how to evaluate competing explanations but since this question does not affect my claim that Frege's argument is not an inductive argument, I assume that some such evaluations can be carried out.[19]

Three characteristics are common to all indispensability argument discussed by Colyvan. One, all these arguments are based on the <u>indispensability</u> of mathematics for the sciences. Two, their main thrust is to support the legitimacy and even the desirability of accepting <u>the existence of mathematical entities</u>, analogously to how other types of entities, which are also indispensable or useful to our best scientific theories such as theoretical entities, are accepted as existent. Finally, all these arguments are ultimately based on <u>an inductive inference</u>. Frege's alleged argument from indispensability differs from the above arguments in lacking these three features.

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[18] See Colyvan (2001), p. 8.

[19] In private correspondence, Resnik suggested that these arguments may be regarded as inductive also because the indispensability of mathematics has been established by many failed efforts of talented logicians to dispense with it. Given how much mathematics is interwoven with theoretical science and how difficult it is to dispense with it, we can inductively infer that we cannot likely dispense with it. Since Colyvan explicitly mentions inference to the best explanation, I discuss only this reason for regarding indispensability arguments as inductive. Yet, since Frege's argument from applicability is not inductive, Resnik's comment provides additional support for my denial that Frege's argument is an indispensability argument.

5. Indispensability and Frege's applicability

Penelope Maddy seems to have been one of the first to claim that "The general idea [of the indispensability argument] traces back at least to Gottlob Frege."[20] Since then, the claim that Frege presents an indispensability argument in the *Grundgesetze* seems to be taken for granted. For example, Mark Balaguer devotes a whole chapter to the "Fregean Argument Against Anti-Platonism,"[21] the second premise of which asserts that "[T]he only way to account for the fact that our mathematical theories are applicable and/or indispensable to empirical science is to admit that these theories are true."[22] Balaguer explicitly acknowledges that his statement of the argument is different from Frege's and points out in a footnote that this latter appeals to the applicability rather than to the indispensability of mathematics; however, he also claims that "the spirit of the argument is essentially the same."[23] Colyvan expresses an analogous point of view:

> The use of indispensability arguments for defending mathematical realism is usually associated with Quine ... but it's important to realize that the argument goes back much further. Gottlob Frege, for example, considers the difference between games such as chess and arithmetic and concludes that "it is applicability alone which elevates arithmetic from a game to the rank of a science." ... As Michael Dummett points out (1991, p. 60), Frege's appeal to the applications of arithmetic here is made in order to raise a problem for formalists who liken mathematics to a game in which mathematical

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[20] See Maddy (1992), p. 275.

[21] See Balaguer (1998), Chapter 5, pp. 95-112.

[22] See Balaguer (1998), p. 95.

[23] See Balaguer (1998), p. 95.

> symbols have no meaning, but are simply manipulated in accordance with certain rules. Frege asks the formalists to explain how such game could have applications. ... This is clearly a form of indispensability argument.[24]

Three parts of this passage deserve comment. First, the quote from Frege's *Grundgesetze* is the one most often cited when the indispensability argument is attributed to Frege. Two, Dummett's remarks point out the purpose of Frege's appeal to the concrete applications of mathematics within his attack against the formalist account of mathematics. Finally, Colyvan draws an analogy between indispensability arguments and Frege's argument based on Frege's request that the formalists *explain* how mathematics could be applied in their account since it is wholly similar to a game like chess and its symbols are devoid of meaning and manipulated according to arbitrary rules. I will come back later to the very useful comment by Dummett. Now, a closer analysis of the first and third aspects will allow me to present my reasons for rejecting Colyvan's claim of an essential similarity between Frege's argument and the indispensability argument.

If the only textual basis for the attribution of an indispensability argument to Frege consists in his claim that "it is applicability alone which elevates arithmetic from a game to the rank of a science," then the similarity between indispensability arguments and Frege's argument is very limited. In this claim there is no mention either of pragmatic or semantic reasons in support of an ontological commitment nor are quantification over mathematical objects and indispensable true mathematical propositions ever mentioned. Furthermore, even appealing to Colyvan's more intuitive notion of indispensability, there is no allusion to a comparison between the indispensability of scientific entities and that of mathematical entities to the relevant scientific theories. Finally, for Frege, the potential *applicability* of mathematics, not its

[24] See Colyvan (2001), p. 8-9.

indispensability to the sciences, elevates mathematics to the rank of a science and makes it so different from chess. This difference though occasionally noted has not been fully appreciated.

To assert that certain mathematical statements are indispensable to the development of a scientific theory is not equivalent to asserting that those statements can be applied within such a theory. If a mathematical statement is indispensable, it is undoubtedly applicable for it could not be indispensable without being actually applied and from its actual application trivially follows its potential applicability. Conversely, however, the mere applicability of a mathematical statement does not ensure its indispensability. Indeed a statement can be applicable, and even be actually applied, without being indispensable; think of useful but not indispensable tools. So, while the indispensability of a mathematical statement is sufficient for its applicability, the latter is not sufficient for the former; a mathematical truth may be applicable without being indispensable. Since indispensability is a stronger property than applicability, Frege's argument and the indispensability arguments appeal to distinct properties. Thus, Frege's argument does not share the first feature of Colyvan's arguments. I now focus on the second of these features.[25]

6. The conclusion and the form of Frege's argument

It is important to read Frege's applicability argument in its textual context. From section 86 to section 137 of the second volume of the *Grundgesetze*, Frege is engaged in a critical

[25] Is the asymmetry between applicability and indispensability relevant to the goal of supporting mathematical realism? *Prima facie*, it may seem to favor Frege's argument for this latter supports realism on the basis of a weaker notion. For example, Field's objection to the indispensability argument does not defeat Frege's argument. If Field's program is successful, it defeats indispensability arguments by denying that mathematical statements are indispensable. Field's objection is irrelevant to Frege's argument since applicability is not a sufficient condition for indispensability and denying the latter does not imply denying the former.

discussion of the theories of two formalists, E. Heine and J. Thomae. Frege argues that these theories propose an internally inconsistent, and thereby inadequate, conception of mathematics:

> [According to Thomae] Arithmetic is concerned only with the rules governing the manipulation of the arithmetical signs, not, however, with the reference of the signs. ... [Thomae] has no consistent terminology. ... [S]ome future historians may be able to show lack of consistency and thoroughness on both sides [i.e., both in Thomae and Heine].
>
> Thomae's admission that even in formal arithmetic numerical signs are not used simply as figures is damaging to his doctrine. For he thereby admits that the formal standpoint cannot be always consistently affirmed.
>
> The assertion that formal arithmetic permits of a completely, consistent foundation accordingly lacks proof; on the contrary, its truth is subject to grave doubts.
>
> This attempt at formal arithmetic must be considered a failure, since it cannot be pursued consistently.[26]

Frege charges the formalists of falling into contradiction; by their theories, numerical terms are to be treated as meaningless figures, but instead they often treat then as signs endowed with reference. For Frege, the formalists often confuse mathematical signs with what they represent.

Within this overarching goal, section 89 opens with the query of what might be the reason for adopting a formalist conception. Frege surmises that, in answer to one such query, Thomae would claim that "[T]he formal standpoint rids us of all metaphysical difficulties;" which would presumably ensue from the ascription of a meaning to mathematical signs. Frege goes on:

[26] See Frege (1960), # 88, p. 192; # 97, p. 192; # 119, p. 215; # 137, p. 232.

> Thomae ...contrasts the arbitrary rules of chess with the rules of arithmetic, the latter causing numbers to make substantial contributions to our knowledge of nature. But this contrast first arises when the applications of arithmetic are in question, when we leave the domain of formal arithmetic. If we stay within its boundaries, its rules appear as arbitrary as those of chess. This applicability cannot be an accident--but in formal arithmetic we absolve ourselves from accounting for one choice of the rules rather than another.[27]

The applicability of mathematics is central to Frege's critique of the formalists. These deny that mathematical terms have reference; as meaningless symbols they are manipulated according to arbitrary rules regardless of their usefulness to any practical goal. The peculiarity of formalism was obvious but could not by itself count against it. Thus, Frege sets out to argue that there is a strong connection, indeed an essential one, between mathematics and its applicability. I submit that this is the conclusion Frege intends to support with his applicability argument.

Colyvan mentions Dummett in support of the thesis that Frege appeals to the applicability of mathematics to attack the formalist conception. In the same page cited by Colyvan, Dummett analyzes Frege's attitude toward concrete employments of arithmetic:

> It is when he is criticising empiricism that Frege insists on the gulf between the senses of mathematical propositions and their applications; it is when he criticises formalism that he stresses that applicability is essential to mathematics.[28]

Dummett sees no conflict between the dismissive attitude Frege displays toward the concrete applications of mathematics while attacking Mill's empiricism and the appreciative attitude he displays toward them when criticizing formalism. Rather, there is a difference of emphasis due

[27] See Frege (1960), p. 185.

[28] See Dummett (1991), p, 60.

to the distinct purposes of Frege's critiques. Dummett points out very clearly the main thrust of Frege's stress on the applicability of mathematics within his attack against the formalists:

> It is not enough that [the fundamental arithmetical notions] be defined in such a way that the possibility of these applications is subsequently provable; since their capacity to be applied in these ways is of their essence, the definitions must be so framed as to display that capacity explicitly.[29]

According to Frege, its applicability is such an essential feature of mathematics that the very definitions of its fundamental notions must clearly show their capability to be applied.

Resnik agrees with Dummett in reading Frege's argument in the *Grundgesetze* as stating a "general philosophical objection" against formalism, i.e., "arithmetic can be applied, whereas meaningless games cannot; so arithmetic is not a meaningless game."[30]

Dummett and Renik each point out an important component of Frege's argument, the essential link between mathematics and its applicability, and the fact that arithmetical propositions are endowed with senses, respectively. By joining these two components, we obtain the conclusion of Frege's argument; the applicability of mathematics is both a warrant and a consequence of the meaningfulness of mathematical statements. Within his polemic against formalism, Frege contrasts formal arithmetic, i.e., the account of arithmetic proposed by formalists such as Heine and Thomae, and meaningful arithmetic, i.e., "that one concerned with actual numbers,"[31] supported for example by Cantor. For Frege, in meaningful arithmetic, "equations and inequations are sentences expressing thoughts," whereas "in formal arithmetic,

[29] See Dummett (1991), p. 60.

[30] See Resnik (1980), p. 62.

[31] See Frege (1960), p. 184.

they are comparable with the positions of chess pieces, transformed in accordance with certain rules without consideration for any sense."[32] If the fomalists were correct, Frege argues, arithmetical equations such as '2 + 2 = 4' would be analogous to positions of chess pieces. Then, similarly to how the rules of chess allow to transform a certain position into another, the rules of mathematics would allow the transformation of an equation into another, e. g., the transformation of the above equation into '2 + 2 = 1 + 3.' For the formalists, the rules of arithmetic are as arbitrary as the rules of chess, exactly because equations are regarded as devoid of sense:

> For if they were viewed as having a sense, the rules could not be arbitrarily stipulated; they would have to be so chosen that from formulas expressing true propositions could be derived only formulas likewise expressing true propositions.[33]

Frege argues: if arithmetical sentences are endowed with a sense, then the rules of mathematics cannot be arbitrary, because they must legitimize only inferences from and to true propositions. Because of the goal of allowing inferences between propositions endowed with a sense, i.e., that express thoughts, Frege rejects an analogy between the rules of chess and the rules of arithmetic.

Here, Frege introduces the notion of the applicability of mathematics. He asks: "Why can no applications be made of a configuration of chess pieces?" and he replies: "Obviously, because it expresses no thought." And again he claims: "an arithmetic with no thought as its content will also be without possibility of application." These are two statements of the same thesis: the expression of a content of thought is a necessary condition for the applicability of a mathematical statement, or else, if a mathematical statement is applicable, then it must be meaningful, i.e., capable of expressing a thought. Yet, this conditional is only a portion of Frege's argument:

[32] See Frege (1960), p. 186.

[33] See Frege (1960), p. 186.

> Why can no application be made of a configuration of chess pieces? Obviously, because it expresses no thought. If it did so and every chess move conforming to the rules corresponded to a transition from one thought to another, applications of chess would also be conceivable. Why can arithmetical equations be applied? Only because they express thoughts. How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules? Now, it is applicability alone which elevates arithmetic from a game to the rank of a science. So applicability necessarily belongs to it. Is it good, then, to exclude from arithmetic what it needs in order to be a science?[34]

Frege advances a stronger conclusion than the mere claim that applicability is a sufficient condition for mathematical statements to be endowed with senses. At least once, he advances the converse and more controversial claim that if a statement expresses a content of thought, then it is applicable, or, in other words, that the applicability of a mathematical statement is a necessary condition for its ability to express a content of thought. For, if the positions of the chess pieces expressed some thought, if "every chess move conforming to the rules corresponded to a transition from one thought to another, applications of chess would also be conceivable." If there were a correspondence between some alterations of the positions of the chess pieces and some transitions from true thoughts to other true thoughts, then we might also find some employment of the positions of the chess pieces. Thus, for Frege, the fact that a statement or an arrangement of pieces represents true thoughts ensures the applicability of that statement or arrangement.

[34] See Frege (1960), p. 187.

Applicability is not only sufficient, but both necessary and sufficient for mathematical statements to express transitions between thoughts, i.e., for a meaningful account of mathematics.[35]

If my reading of Frege's argument is correct, Frege is not arguing for the truth of mathematical statements or the existence of mathematical entities, but for the thesis that in order to provide an accurate account of mathematics, we need to account for its applicability. Applicability is an essential feature of mathematics, which entails its meaningfulness, and in turn, its being a science and not a mere game. Hence, Frege's applicability argument does not share the second feature of Colyvan's indispensability arguments.

This discussion supports also the denial that Frege's argument displays the third feature of Colyvan's arguments. Frege's intent is to show that the applicability of mathematics is an

[35] A footnote at the end Frege's discussion of the formalist theories might seem in conflict with the interpretation of Frege's views proposed in this essay: "As though the question about the truth of a thought and its applicability were not quite different? I can very well recognize the truth of a proposition without knowing whether I will ever have a chance to make use of it." See Frege (1960), p. 233, footnote 2. The first sentence seems to deny a connection between the applicability and the truth of a thought. I have two comments on this. One, in the context of this footnote, in which he is attacking the views of H. v. Helmholtz, another formalist, Frege objects to the empiricist overtones of Helmholtz' views. Just before what quoted above, he says: "To add to the confusion, psychology and empiricism are dragged in. Helmholtz is out to found arithmetic empirically, whether it bends or breaks. ... All who have this desire succeed very easily by confusing the application of arithmetical theorems with the theorems themselves." Here, Frege's preoccupation is best described by Dummett's comment: "[W]hen he is criticizing empiricism ... Frege insists on the gulf between the senses of mathematical propositions and their applications." See Dummett (1991), p. 60. Two, in the second sentence of the first quote from the footnote, Frege distinguishes between our abilities to recognize the truth of a statement and our abilities to know an application for it. Denying a connection between our knowledge of a truth and our knowledge of some application of that truth is not the same as denying a connection between the meaningfulness or truth of a mathematical statement and its applicability.

essential component of it; hence, one that allows us to show that mathematical statements must be meaningful. In support of this strong connection, Frege does not rely on inductive reasoning. In his argument, there is no hint of an inference to the best explanation, nor of any induction based on scientists's efforts to dispense with mathematical statements, theories, or objects. Thus, also the third potential reason for an analogy between Frege's argument and Colyvan's fails.

7. Conclusion

A supporter of Colyvan's comparison might claim an analogy between Putnam's or Quine's conditional "if mathematical statements are indispensable, then there exist entities described by them," and Frege's conditional: "if mathematical statements are applicable, then they express thoughts." The contrast between Frege's assertion of the existence of thoughts and Quine's and Putnam's assertion of the existence of mathematical entities is interesting, but it is ultimately irrelevant to my conclusion. If these are distinct realist theses, they are supported by different arguments. On the other side, if these are comparable forms of realism, an argument is due to support this analogy; so far, neither Colyvan nor anyone else who claims that Frege has advanced an indispensability argument has yet given such an argument. I do not deny that it might be possible to reconstruct a Fregean indispensability argument on the basis of other texts by Frege, but only that Frege does not present one such argument in section 91. [36]

[36] It would be worthwhile to compare Frege's argument and indispensability arguments as to their ability to support mathematical realism and to discuss whether or not Frege connected the meaningfulness of numerical terms, their denotative power, with realism. I thank an anonymous referee for this journal for these well focused suggestions for further inquiry.

In that section, Frege denies that an account of mathematics such as the formalists', which did not explain the capacity of mathematical statements to be applied in concrete cases, can fully capture one essential feature of mathematics, i.e., the link between its applicability and the power of its symbols and its laws to express transitions between true thoughts. For Frege, exactly this link makes mathematics different from games like chess and similar to science.[37]

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